Climb of a bore on a beach

Part 2. Non-uniform beach slope

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The investigation of the shoreward travel of a bore into water at rest is extended to beaches of non-uniform slope. It is shown that the shore singularity established in Part 1 (Ho & Meyer 1962) still furnishes an approximate solution. The shape of the beach, like the shape of the wave forming the bore, is found to influence the bore development close to shore almost only in so far as it determines the basic velocity scale of the bore.

1. Introduction

The investigation of the climb of a bore into water at rest on a beach reported in Part 1 (Ho & Meyer 1962) has led to an understanding of the shore singularity for a fairly large range of boundary conditions, but only for beaches of uniform slope. Such beaches are mathematically exceptional in that a transformation of the non-linear shallow-water equations into the linear Euler–Poisson–Darboux equation is possible *only* for them. It will be shown now that the main results regarding the shore singularity are not similarly exceptional.

The support which this gives to the work reported in Part 1 appears desirable, in view of the somewhat unconventional approach there employed and the quite heavy reliance there placed on the mathematical simplification afforded by uniformity of the beach slope. Actually, the arguments of Part 1 do not rely directly on linearity and superposition, since the approach is based mainly on the theory of the structure of the non-singular Euler–Poisson–Darboux equation. But much as it be plausible that the basic structure of the solutions depend little on the uniformity of beach slope, the canonical equations, from which the structure is deduced, are doubled in number, and made more complicated in form, by abandonment of the uniformity of beach slope. Accordingly, bridging the gap between the seaward boundary condition and the shore irregularity becomes a more formidable task.

It is not, in fact, attempted here. Rather, it will be shown in § 2 that some basic conclusions of Part 1 depend only on the assumption that the bore reaches the shore at a finite time. Assuming the beach slope to be finite and non-zero at the initial shore position, and the beach curvature to be continuous in a neighbourhood of that position, or at any rate, not too strongly singular there, it will then be deduced in § 3 that the solution of Part 1 is still an approximate solution near the shore. It furnishes again a very detailed asymptotic description of the bore development, indicating that all wave shapes and all beach shapes—within quite a wide range—lead effectively to only a two-parameter family of bore developments near the shore. These two parameters are the beach slope at the initial shore position and the basic velocity scale of the motion.

2. Governing equations

The shoreward propagation of a straight bore into undisturbed water of depth $h_0(x)$ has been studied in Part 1† on the basis of the non-linear, first-order shallow-water theory (Stoker 1957). The differential equations of the water motion and the bore relations are respectively given by (I.2), (I.3) and (I.4) to (I.6). The properties of the solutions obtained in Part 1 were derived only for the particular case of constant beach slope. In the following, it will be assumed that the beach slope

$$dh_0/dx = -\gamma(x)/g \neq \text{const.}, \tag{1}$$

with $\gamma(x)$ continuous and $\gamma(0) = \gamma_0 > 0$. The horizontal distance x will again be measured landward from the initial shore position, so that $h_0(0) = 0$, and since our concern is with the neighbourhood of the initial shore position, no generality is lost in taking $h_0(x)$ to be a monotonically decreasing function and

$$\delta(x) = \gamma(x) - \gamma_0$$

to be non-zero for $x < 0$.

Recall that u denotes the water velocity in the direction of x increasing (figure I.1) and $h(x, t)$, the disturbed water depth. Let

$$c^2 = gh(x, t) \quad (c \geqslant 0), \tag{2}$$

and

$$2c + u + \gamma_0 t - u_0 = \alpha, \qquad 2c - u - \gamma_0 t + u_0 = \beta, \tag{3}$$

where u_0 is a constant. The shallow-water equations are hyperbolic, and if ξ, η denote parameters labelling respectively the advancing and receding characteristics in a $1-1$ manner, (I.2, 3) may be written

$$\frac{\partial x}{\partial \eta} = (u + c)\frac{\partial t}{\partial \eta}, \qquad \frac{\partial \alpha}{\partial \eta} + \delta(x)\frac{\partial t}{\partial \eta} = 0, \tag{4}$$

$$\frac{\partial x}{\partial \xi} = (u - c)\frac{\partial t}{\partial \xi}, \qquad \frac{\partial \beta}{\partial \xi} - \delta(x)\frac{\partial t}{\partial \xi} = 0. \tag{5}$$

In addition to the bore relations, as in Part 1, some boundary conditions are required on a seaward boundary which may be taken (§I.7) to be a segment either of a line $x = \text{const.} < 0$ or of a receding characteristic line C (figure 1).

We now introduce the assumption that the bore reaches the shore at a finite time, again chosen as $t = 0$. This implies (§I.3) both the relations

$$c_b \to 0 \quad \text{as} \quad t \to 0, \tag{6}$$

$$h_b > h_0 \quad \text{for} \quad t < 0, \tag{7}$$

$$u_b + c_b > v_b > c_b > 0, \qquad v_b > u_b > 0 \quad \text{for} \quad t < 0, \tag{8}$$

† Ho & Meyer (1962). Equation, figure and section number of Part 1 will be distinguished by a prefix I.

and the existence of a limiting characteristic L. Moreover, it implies (§ I.3) that we may, by reducing the time interval $T_0 \leqslant t \leqslant 0$ during which the bore development is studied, ensure the absence of any secondary bores from the region II (figure 1) bounded by the bore B, limiting characteristic L and seaward boundary C. Then α and β are continuous functions of time on the seaward boundary, and

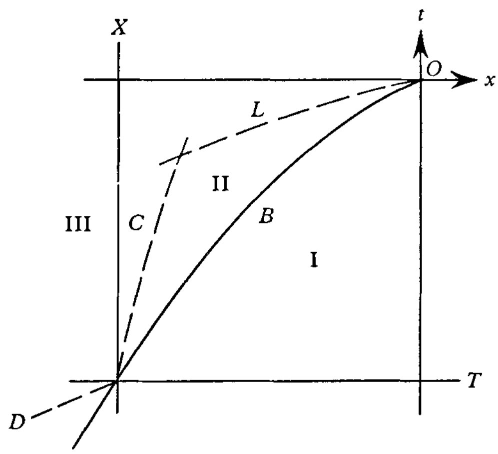

FIGURE 1. Diagram of (x, t)-plane showing locus of successive bore positions (bore initially supercritical).

the solution of (4), (5) must be continuous in the interior of region II. Since $\delta(x)$ is continuous, the uniform continuity of the solution in region II follows from the second of (4) and from (3), which show that u_b tends to a finite limit as $t \to 0$. The constant u_0 of (3) will again be identified with this limit, so that

$$\alpha \to 0, \quad \beta \to 0 \quad \text{as} \quad t \to 0 \quad \text{on} \quad L, \tag{9}$$

$$u_b \to u_0, \quad \alpha_b \to 0, \quad \beta_b \to 0 \quad \text{as} \quad t \to 0 \quad \text{on} \quad B, \tag{10}$$

and from (I.5, 6), h_b and h_0 must vanish together.

We shall also assume that physically consistent seaward boundary conditions are set, and as in Part 1, the absence of secondary bores then implies (Mahony 1956) the absence of limit points $\partial t/\partial \xi = 0$ or $\partial t/\partial \eta = 0$ in the interior of region II.

We may consider x, t and u as functions of α and β, and then (5) and (6) become formally

$$\left.\begin{aligned} [x_\alpha - (u+c)\,t_\alpha]\,\partial\alpha/\partial\eta + [x_\beta - (u+c)\,t_\beta]\,\partial\beta/\partial\eta &= 0,\\ (1+\delta t_\alpha)\,\partial\alpha/\partial\eta + \delta t_\beta\,\partial\beta/\partial\eta &= 0,\end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} [x_\alpha - (u-c)\,t_\alpha]\,\partial\alpha/\partial\xi + [x_\beta - (u-c)\,t_\beta]\,\partial\beta/\partial\xi &= 0,\\ \delta t_\alpha\,\partial\alpha/\partial\xi - (1-\delta t_\beta)\,\partial\beta/\partial\xi &= 0,\end{aligned}\right\} \tag{12}$$

where suffixes denote partial differentiation. Now, $\delta(x)$, $\partial t/\partial \xi$ and $\partial t/\partial \eta$ are nonzero in the interior of region II (figure 1), and the same follows for $\partial\alpha/\partial\eta$ and $\partial\beta/\partial\xi$ from (4) and (5). The necessary and sufficient conditions for (11) and (12) to possess such solutions are respectively

$$x_\beta - (u+c)\,t_\beta = \delta j, \tag{13}$$

$$x_\alpha - (u-c)\, t_\alpha = \delta j, \tag{14}$$

where
$$j = x_x t_\beta - x_\beta t_\alpha = \partial(x,t)/\partial(\alpha,\beta). \tag{15}$$

3. An approximate solution

The problem at hand is to solve (13), (14) together with the bore relations and seaward boundary conditions as in Part 1. The dominant feature of the differential equations, however, is the same shore-singularity which also dominates them for a beach of uniform slope, and it is therefore plausible that the solution should again be dependent on the seaward boundary conditions mainly because they determine the limit velocity u_0. A quantitative analysis of the dependence of u_0 on these boundary conditions is still outstanding even for the uniformly sloping beach. An analysis of the dependence of u_0 on beach slope variations therefore appears premature, and we shall limit ourselves to showing that (13), (14) and the bore conditions can be satisfied asymptotically by an approximation possessing the type of shore singularity derived in Part 1. This will enable us to confirm the existence of an asymptotic approximation which, apart from the basic velocity scale u_0, depends only very weakly on beach slope variations.

Part 1 is devoted to the discussion of a set of functions $x = X(\alpha,\beta)$, $t = T(\alpha,\beta)$, $c = (\alpha+\beta)/4$, $u = U(\alpha,\beta) = u_0 - \gamma_0 T + (\alpha-\beta)/2$, $v_b = V(\beta)$ and $\alpha_b = A_b(\beta)$ which satisfy
$$X_\alpha = (U-c)\, T_\alpha, \quad X_\beta = (U+c)\, T_\beta. \tag{16}$$

In the region II of figures 1 and I.3, i.e. for $A_b \geqslant \alpha > 0$ and $\beta > 0$, these functions have the properties that (I)

$$X = O(c^4), \quad U = u_0 - 2c + O(c^4), \quad V - U_b = O(c^2), \tag{17}$$

$$T_\alpha = O(c^{-\frac{3}{2}}), \quad T_\beta = O(c^3), \quad A_b = O(c^{\frac{11}{2}}), \tag{18}$$

as $c \to 0$. From (16), therefore,

$$J = \partial(X,T)/\partial(\alpha,\beta) = -2cT_\alpha T_\beta = O(c^{\frac{5}{2}}), \tag{19}$$

while $(U+c)\, T_\beta = O(c^3)$ and $(U-c)\, T_\alpha = O(c^{-\frac{3}{2}})$. Thus δJ becomes negligible in comparison with both the left-hand sides and right-hand sides of (16) and hence, the functions X, T, U and c satisfy (13) and (14) to a first approximation, provided $\delta(x) = o(x^{\frac{1}{8}})$.

By (13) and (14), the bore condition $dx_b/dt_b = v_b$ is equivalent to

$$(u - v_b - c)\, t_\alpha\, d\alpha_b/d\beta + (u - v_b + c)\, t_\beta = -(1 + d\alpha_b/d\beta)\, \delta j \quad \text{on} \quad \alpha = \alpha_b(\beta), \tag{20}$$

while (I)

$$(U - V - c)\, T_\alpha\, dA_b/d\beta + (U - V + c)\, T_\beta = 0 \quad \text{on} \quad \alpha = A_b(\beta). \tag{21}$$

From (17) and (18), $(U - V + c)\, T_\beta = O(c^4)$, and so by (19), U, V, c, T and A_b satisfy (20) asymptotically, if $\delta = o(x^{\frac{3}{8}})$. The other bore conditions (I.5, 6)

$$u_b/v_b = 1 - h_0(x_b)/h_b, \quad 2v_b^2 = gh_b\{1 + h_b/h_0(x_b)\}, \tag{22}$$

do not involve x, t or their derivatives directly and, due to the continuity of $h_0(x)$, are obviously satisfied asymptotically. In short, the asymptotic solution of Part 1 satisfies the shallow-water equations and bore conditions asymptotically to a first approximation even on a beach of non-uniform slope, provided only that the beach curvature does not possess too strong a singularity at the initial shore position. This approximation, moreover, has the properties

$$\left.\begin{aligned} x &= O(c^4), \quad u = u_0 - 2c + O(c^4), \quad v_b - u_b = O(c^2), \\ t_\alpha &= O(c^{-\frac{3}{2}}), \quad t_\beta = O(c^3), \quad \alpha_b = O(c^{\frac{11}{2}}). \end{aligned}\right\} \tag{23}$$

It may be worth noting that any more detailed description of the water acceleration must generally be expected to depend on the beach slope variation. From I, a number $a_0 > 0$ exists such that

$$(\alpha+\beta)^{\frac{3}{2}} T_\alpha - a_0 = O(c^{\frac{9}{2}}),$$

but comparison of (20) and (21) shows that an asymptotic solution with the property

$$(\alpha+\beta)^{\frac{3}{2}} t_\alpha - a_0 = O(c^{\frac{9}{2}})$$

can be anticipated only if $\delta = O(x^{\frac{3}{2}})$, which appears too stringent a condition for most practical cases.

None the less, a very detailed asymptotic description of the bore development can be obtained from the first approximation, much as in Part 1. Let

$$z = c_b/(u_b + 2c_b), \qquad M = v_b/c_b, \qquad H = h_0(x_b)/h_b, \tag{24}$$

then (I) the bore conditions (22) give M and H in terms of z by

$$z^{-1}(1-2z) = u_b/c_b = M(1-H), \qquad M^2 = (1+H^{-1})/2 \tag{25}$$

and by (1) and (23) to (25),

$$\gamma_0 t_b = -\int \frac{\gamma_0}{\gamma v_b}\, d(Hc_b^2) = -\frac{\gamma_0 H}{\gamma M} c_b + \tfrac{1}{5} u_0 z^5 + o(z^5). \tag{26}$$

Since $\gamma_0/\gamma = 1 + O(\delta)$, (3) now gives (if $\lim (\alpha+\beta)^{\frac{3}{2}} t_\alpha$ be again denoted by a_0)

$$c_b/u_0 = z(1 - Hz/M)^{-1} [1 - \tfrac{1}{5} z^5 - \tfrac{16}{11} (\gamma_0 a_0)^{-1} u_0^{\frac{3}{2}} z^{\frac{11}{2}} + o(z^{\frac{11}{2}})], \tag{27}$$

which is identical with (I.48), provided $\delta = o(x^{\frac{3}{8}})$. By (24) and (25), the asymptotic approximations obtained for v_b/u_0, u_b/u_0 and $gh_0(x_b)/u_0^2$ are also the same as those derived in Part 1, and the main influence of beach slope variation on the asymptotic behaviour of all these variables is thus seen to be exerted 'indirectly' through their influence on u_0 and a_0. The relation between x_b and $h_0(x_b)$, however, differs from that envisaged in Part 1; in the most common case of non-zero, finite beach curvature, $\delta = O(x) = O(c^4)$, so that a term $O(c_b^8)$ depending on δ appears in the approximation for x_b. A term of the same order appears in that for t_b, due to the factor γ_0/γ in (26), and these terms are large compared with those depending on a_0 in the respective approximations (I).

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